ON THE MINIMAL CYCLE LENGTHS OF THE COLLATZ SEQUENCES

MATTI K. SINISALO

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Department of Mathematical Sciences
University of Oulu
SF-90570 OULU
FINLAND

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Matti K. Sinisalo

1§. Introduction

Let x be an integer. Let the function $C(x)$ be equal to $3x+1$ if x is odd and equal to $x/2$ if x is even. Iterating this function from the initial value 1 we get a trivial cycle $\{1,4,2\}$. The $3x+1$ conjecture asserts that starting from any positive integer a_1 the repeated iteration of $C(x)$ eventually produces the integer 1, after which the iterates will alternate between the integers of the trivial cycle. This assertion posed by L. Collatz in 1937 has turned out to be very hard to prove. It has been widely studied in different places all over the world. Besides the Collatz problem, it has been called by many other names, like $3x+1$ mapping, Hasse's algorithm, Kakutani's problem, Syracuse algorithm or problem, Thwaites conjecture, and Ulam's problem. In spite of massive try and effort, the Collatz conjecture is still unproven.

Number a_1 is said to *converge* if $C^h(a_1) = 1$ for some positive integer h. In order to prove that all positive initial values a_1 up to some number R converge, it is sufficient to check that for all $2 < a_1 \leq R$ there exist some integer h such that $C^h(a_1) < a_1$. If Collatz conjecture is not true for some integer a_1 we have two possibilities: Either 1) the sequence starting from a_1 turns to infinity or 2) the sequence enters to cycle other than the trivial one $\{1, 2, 4\}$.

Until now all numbers up to $204 * 2^{50}$ $(\approx 2.29 * 10^{17})$ have been checked for convergence. All numbers up to this bound have been verified to converge **[7]**. In this paper we are going to study possible nontrivial cycles in Collatz sequences. If there exists a nontrivial cycle we know that every integer belonging to this cycle must be larger than the bound referred to above.

Lagarias showed in 1985 that there are no nontrivial cycles with length less than *275,000*.

It should be noted that Lagarias used $(3x+1)/2$ operation instead of $3x+1$ operation. This gives shorter cycle lengths. We will return to this later.

Our aim in this paper is to show that there are no nontrivial cycles with length less than *1,000,000,000*. We also show that only certain discrete values are possible for the cycle lengths of the Collatz sequence.

2§. Basic inequalities

LEMMA 1: *Let us suppose that there exists an untrivial cycle of length m in some Collatz sequence with positive initial value. Let k be the number of 3x+1 operations and n the number of x/2 operations in the cycle. Hence k+n=m. Suppose that the smallest number in the sequence is greater than some (with a computer reachable bound) number R. Then* $\ln(3)/\ln(2) < n/k \le \ln(3+1/R)/\ln(2)$.

PROOF: Let $c_1, c_2,\ldots,c_m$ be the rational integer numbers of the cycle i.e. $c_{i+1} = C(c_i)$ for all $i=1,2,\ldots,m-1$ and $c_1 = C(c_m)$. Without restricting the generality we can assume c_1 to be the smallest one of them.

Let $r_1, r_2,\ldots r_k$ be the ratios of succeeding integers in sequence when $3x+1$ operation has been used. Now $r_1 r_2 \ldots r_k = 2^n$.

On the other hand $3 < r_i \le (3c_1+1)/c_1 = 3+1/c_1$ for all $i=1,\ldots,k$.

Hence $3^k < 2^n = r_1 r_2 \ldots r_k \le (3+1/c_1)^k$, and therefore $k \ln(3) < n \ln(2) \le k \ln(3+1/c_1)$.

Consequently

$$\frac{\ln(3)}{\ln(2)} < \frac{n}{k} \le \frac{\ln(3+\frac{1}{c_1})}{\ln(2)} .$$

If $R \le c_1$ we have

$$\frac{\ln(3)}{\ln(2)} < \frac{n}{k} \le \frac{\ln(3+\frac{1}{R})}{\ln(2)} . \square$$

Our problem now is to find the rational number n/k such that these inequalities are satisfied and $n+k$ is as small as possible.

We will show that this follows if the denominator k is the least possible. For this we need some results concerning rational approximations.

3§. Results concerning Farey sequences and rational approximations

By the Farey sequence F_m of order m we mean the positive fractional numbers, whose denominators do not exceed m, arranged in ascending order of magnitude.

Let us take some notations and terms in use. If x is a real number, then $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. Subsequently we define residual function Mod$:Z \times Z^+ \to Z$ by setting $m = \lfloor m/n \rfloor\ n + \mathrm{Mod}(m,n)$ for all $m \in Z$, $n \in Z^+$.

If m and n are rational integers, $n > 0$ and $\gcd(m,n) = 1$, then using (extended) Euclidean algorithm we can find such rational integers a and b, that $am+bn = 1$. Then $am \equiv 1 \pmod{n}$. Congruence equation $mx \equiv 1 \pmod{n}$ can hence be effectively solved. This solution is called the *modular inverse* of the number m modulo n.

A rational number is in *reduced form* if the greatest common divisor of the numerator and the denominator is 1.

When processing with Farey sequences the following lemma is essential, see [4].

LEMMA 2: *Let p, q and m be positive integers, $q \le m$ and $\gcd(p,q) = 1$. Let r be a modular inverse of p modulo q and*

$$q' = \left\lfloor \frac{m+r}{q} \right\rfloor q - r = m - \mathrm{Mod}(m+r,q) \quad \text{and} \quad p' = \frac{pq'+1}{q}.$$

Then $0 < q' \le m$, $\gcd(p',q')=1$ and the number p'/q' is the smallest rational number greater than p/q and with the denominator $\le m$.

PROOF: 1) Clearly q' is a rational integer. Consequently $pq'+1 = p\lfloor (m+r)/q \rfloor q - pr + 1 \equiv 0 \pmod{q}$. Hence p' is also a rational integer.

2) Because of $p'q - pq' = 1$ we have $\gcd(p',q') = 1$. Hence number p'/q' is in reduced form.

3) Multiplying the inequalities

$$0 \le \frac{m+r}{q} - \left\lfloor \frac{m+r}{q} \right\rfloor \le \frac{q-1}{q}$$

by the number q and adding number r we get $r \le m+r-q' \le q-1+r$, and consequently $m+1-q \le q' \le m$.

4) We show that between the rational numbers p/q and p'/q' there cannot exist any rational number having denominator $\le m$. Let us suppose that $p/q < s/t < p'/q'$, $0 < t \le m$. Then $sq - pt \ge 1$, $p't - sq' \ge 1$ and consequently

$$1 = p'q - pq' = qq'\left(\frac{p'}{q'} - \frac{p}{q}\right) = qq'\left(\left(\frac{p'}{q'} - \frac{s}{t}\right) + \left(\frac{s}{t} - \frac{p}{q}\right)\right)$$
$$= qq'\left(\frac{p't - sq'}{tq'} + \frac{sq - pt}{tq}\right) \ge qq'\left(\frac{1}{tq'} + \frac{1}{tq}\right) = \frac{q+q'}{t} > \frac{m}{t} \ge 1.$$

This is a contradiction. Hence p/q and p'/q' are successive rational numbers in Farey sequence F_m. □

The essential parts of the proof can be found in the book of Hardy and Wright 1938 [3]. The preceding computationally useful form is not given, anyway.

The following lemma can be proved analogously to Lemma 2.

LEMMA 3: *Let p,q,m be positive rational integers, $q \leq m$ and* gcd*(p,q)=1. Let r be a modular inverse of p modulo q and*

$$q'' = \lfloor \frac{m-r}{q} \rfloor q+r = m\text{-Mod}(m-r,q) \text{ and } p'' = \frac{pq''-1}{q}.$$

Then $0 < q'' < m$, gcd*$(q'',p'')=1$ and the number p''/q'' is the greatest rational number smaller than p/q and with the denominator $\leq m$.*

The preceeding lemmas give explicit expressions to the immediately following and to the immediately preceding numbers of a given number in some Farey sequence. The method is based on the modular arithmetics and the Euclidean algorithm. This method works (by Mathematica experiments) quite well even with the numbers with *1000-10000* decimal digits.

The Farey sequences have applications in a very wide area of mathematics. However, our computationally important explicit results have not been presented, as far as we know, in the litterature of the elementary number theory, computational number theory or the theory of mathematical algorithms.

We can now write short Mathematica programs for using lemmas 2 and 3 effectively.

```
NextFarey[s_, m_] :=
  First[{p = Numerator[s];
        q = Denominator[s];
        r = First[Part[ExtendedGCD[p, q], 2]];
        q2 = Quotient[m + r, q]*q - r;
        p2 = (p*q2 + 1)/q;
        p2/q2}]

PreviousFarey[s_, m_] :=
  First[{p = Numerator[s];
         q = Denominator[s];
         r = First[Part[ExtendedGCD[p, q], 2]];
         q2 = Quotient[m - r, q]*q + r;
         p2 = (p*q2 - 1)/q;
         p2/q2}]

NextBestUpper[s_] :=
    NextFarey[s, Denominator[s]]
```

```
NextBestLower[s_] :=
    PreviousFarey[s, Denominator[s]]
```

In the proof of Lemma 2 we have also proved the following Lemma.

LEMMA 4: *(Farey-Cauchy theorem) If p/q and p'/q', where $0 < q$, $0 < q'$ and* $\gcd(p,q) = \gcd(p',q') = 1$, *are any successive rational numbers in Farey sequence F_m, then $p'q - pq' = 1$.*

Using Farey-Cauchy theorem we can prove the following simple lemma.

LEMMA 5: *Let $0 \leq \lfloor\beta\rfloor < \alpha < n/k \leq \beta$ and let k be the least possible positive denominator value for which these inequalities are satisfied. Then*
a) *$k > 1$, n/k is unique and*
b) *if c/d, where $d > 1$ is any other rational number for which $\alpha < c/d \leq \beta$ then $n+k \leq c+d$.*

PROOF: a) It's clear that $k > 1$. Trivially k has a unique value. If n is not unique, then either 1) $\alpha < j/k < n/k \leq \beta$ or 2) $\alpha < n/k < j/k \leq \beta$ for some positive integer j for which $\gcd(j,k) = 1$.

In case 1) j/k and n/k are successive rational numbers in order k Farey sequence. By the Farey-Cauchy theorem $1 = kn - kj = k(n-j) \geq k$. This is a contradiction.

In case 2) n/k and j/k are successive rational numbers in order k Farey sequence. By the Farey-Cauchy theorem $1 = jk - nk = (j-n)k \geq k$. This is a contradiction.

b) By a) we have $\alpha \geq (n-1)/k$ (n is unique) and by the definition of k, $d \geq k$.

Hence

$$\frac{c}{d} > \alpha \Rightarrow c > \alpha d \geq \frac{n-1}{k}d \Rightarrow ck > (n-1)d \Rightarrow ck \geq (n-1)d + 1 \Rightarrow c \geq \frac{(n-1)d+1}{k}$$

$$\Rightarrow c+d \geq \frac{(n-1)d+1}{k} + d \geq \frac{(n-1)k+1}{k} + k = n - 1 + \frac{1}{k} + k = n + k - 1 + \frac{1}{k} > n + k - 1$$

$$\Rightarrow c + d \geq n + k\,. \square$$

4§. Back to the Collatz problem

We are now ready to prove our main results concerning the cycle lengths of the Collatz sequences.

THEOREM 1: *Let the Collatz conjecture be verified up to some bound $R > 1$. Let n/k be the rational number with least possible denominator k such that* $\ln(3)/\ln(2) < n/k \le \ln(3+1/R)/\ln(2)$. *Then the least possible cycle length for nontrivial cycle is $n+k$.*

PROOF: Let c be the number of $x/2$ operations and d the number of $3x+1$ operations in the shortest nontrivial cycle. By lemma 1 we have $\ln(3)/\ln(2) < c/d \le \ln(3+1/R)/\ln(2)$. Let us denote $\alpha = \ln(3)/\ln(2)$ and $\beta = \ln(3+1/R)/\ln(2)$. Now

$$1 = \left\lfloor \frac{\ln(3)}{\ln(2)} \right\rfloor \le \left\lfloor \frac{\ln(3+\frac{1}{R})}{\ln(2)} \right\rfloor = \lfloor \beta \rfloor \le \frac{\ln(3+\frac{1}{R})}{\ln(2)} < \frac{\ln(4)}{\ln(2)} = 2.$$

Necessarily $\lfloor \beta \rfloor = 1$.

Now $0 < 1 = \lfloor \beta \rfloor < \ln(3)/\ln(2) = \alpha < n/k < \beta$.

By lemma 5 integers n and k are unique and $k > 1$. Furthermore we have $c+d \ge n+k$. But $c + d$ equals to the length of the cycle. □

THEOREM 2: *The length of any nontrivial cycle in Collatz sequence is at least 1,027,712,276.*

PROOF: The 20. convergent of $\ln(3)/\ln(2)$ is the first one of them that satisfies our inequalities when $R = 204*2^{50}$. It's value is *630,138,897/397,573,379*. The closest neighbours of it in order *397,573,379* Farey sequence are *357,638,239/225,644,606* and *272,500,658/171,928,773*. They don't belong to our interval. So our convergent has the least possible denominator in our interval and the sum of its numerator and denominator is the least possible length of a nontrivial cycle in Collatz sequence. Now we get $m = n + k = 630{,}138{,}897 + 397{,}573{,}379 = 1{,}027{,}712{,}276$. □

NOTE 1. Some researchers use $(3x+1)/2$ operation instead of $3x+1$ operation that we have used in the definition of the Collatz sequence. In this case the numbers $3x+1$ and following $(3x+1)/2$ are not considered as different iterates of the cycle. This gives shorter cycle lengths. It is easy to see that instead of the sum of the numerator and the denominator the cycle length is given by just numerator of our rational approximation. When the operations $x/2$ and $(3x+1)/2$ are used the minimal cycle length is hence still at least *630,138,897*.

NOTE 2. Tomas Oliveira e Silva has checked the Collatz sequences to converge with all initial values up to $3*2^{53}$ ($\approx 2.702*10^{16}$) **[8]**. As far as we know, his result is the best one published in international refereed mathematical journals. It is easy to see, that this result gives exactly the same minimal cycle length as we have reached before.

The following table gives best rational upper approximations for $\ln(3)/\ln(2)$. If an approximation is a continued fraction convergent its order number is given in the second column. The third column gives the sum of the numerator and the denominator of the rational approximation. This is the length of representing cycle of Collatz sequence. The last column gives the bound to be reached by computer calculations or by theoretical studies to eliminate the possibility of cycle of this size to exist.

Rational approximation n/k	**Convergent number**	**Cycle length m = n + k**	**Computational bound $R = 1/(2^{n/k}-3)$**
683,381,996,816,440 / 431,166,034,846,567	*30.*	*1,114,548,031,663,007*	1.07756×10^{29}
600,251,839,738,223 / 378,716,745,326,851	-	*978,968,585,065,074*	4.54238×10^{28}
517,121,682,660,006 / 326,267,455,807,135	-	*843,389,138,467,141*	2.57441×10^{28}
433,991,525,581,789 / 273,818,166,287,419	-	*707,809,691,869,208*	1.60979×10^{28}
350,861,368,503,572 / 221,368,876,767,703	-	*572,230,245,271,275*	1.03707×10^{28}
267,731,211,425,355 / 168,919,587,247,987	-	*436,650,798,673,342*	6.57741×10^{27}
184,601,054,347,138 / 116,470,297,728,271	-	*301,071,352,075,409*	3.88004×10^{27}
101,470,897,268,921 / 64,021,008,208,555	-	*165,491,905,477,476*	1.86358×10^{27}
18,340,740,190,704 / 11,571,718,688,839	*28.*	*29,912,458,879,543*	2.99088×10^{26}
8,573,543,875,303 / 5,409,303,924,479	*26.*	*13,982,847,799,782*	2.73493×10^{25}
7,379,891,435,205 / 4,656,193,084,598	-	*12,036,084,519,803*	8.39455×10^{24}
6,186,238,995,107 / 3,903,082,244,717	-	*10,089,321,239,824*	4.28181×10^{24}
4,992,586,555,009 / 3,149,971,404,836	-	*8,142,557,959,845*	2.48336×10^{24}
3,798,934,114,911 / 2,396,860,564,955	-	*6,195,794,679,866*	1.47471×10^{24}
2,605,281,674,813 / 1,643,749,725,074	-	*4,249,031,399,887*	8.29256×10^{23}
1,411,629,234,715 / 890,638,885,193	-	*2,302,268,119,908*	3.80765×10^{23}
217,976,794,617 / 137,528,045,312	*24.*	*355,504,839,929*	5.10126×10^{22}
114,208,327,604 / 72,057,431,991	-	*186,265,759,595*	4.35849×10^{21}
10,439,860,591 / 6,586,818,670	*22.*	*17,026,679,261*	2.16891×10^{20}
630,138,897 / 397,573,379	*20.*	*1,027,712,276*	1.25208×10^{18}
272,500,658 / 171,928,773	*18.*	*444,429,431*	3.20306×10^{16}
187,363,077 / 118,212,940	-	*305,576,017*	7.48875×10^{15}
102,225,496 / 64,497,107	-	*166,722,603*	2.46143×10^{15}
17,087,915 / 10,781,274	*16.*	*27,869,189*	2.94402×10^{14}
301,994 / 190,537	*14.*	*492,531*	9.84573×10^{11}
125,743 / 79,335	*12.*	*205,078*	7.21611×10^{9}
75,235 / 47,468	-	*122,703*	1.44769×10^{9}
24,727 / 15,601	*10.*	*40,328*	2.85818×10^{8}
23,673 / 14,936	-	*38,609*	8.04976×10^{7}
22,619 / 14,271	-	*36,890*	4.50889×10^{7}
21,565 / 13,606	-	*35,171*	3.04065×10^{7}
20,511 / 12,941	-	*33,452*	2.23726×10^{7}
19,457 / 12,276	-	*31,733*	1.73049×10^{7}

18,403 / 11,611	-	*30,014*	1.38168×10^{7}
17,349 / 10,946	-	*28,295*	1.12692×10^{7}
16,295 / 10,281	-	*26,576*	9.32702×10^{6}
15,241 / 9,616	-	*24,857*	7.79732×10^{6}
14,187 / 8,951	-	*23,138*	6.56133×10^{6}
13,133 / 8,286	-	*21,419*	5.54185×10^{6}
12,079 / 7,621	-	*19,700*	4.68659×10^{6}
11,025 / 6,956	-	*17,981*	3.95881×10^{6}
9,971 / 6,291	-	*16,262*	3.33200×10^{6}
8,917 / 5,626	-	*14,543*	2.78651×10^{6}
7,863 / 4,961	-	*12,824*	2.30747×10^{6}
6,809 / 4,296	-	*11,105*	1.88344×10^{6}
5,755 / 3,631	-	*9,386*	1.50545×10^{6}
4,701 / 2,966	-	*7,667*	1.16640×10^{6}
3,647 / 2,301	-	*5,948*	*860,564*
2,593 / 1,636	-	*4,229*	*583,288*
1,539 / 971	-	*2,510*	*330,750*
485 / 306	*8.*	*791*	*99,781*
401 / 253	-	*654*	*27,114*
317 / 200	-	*517*	*12,825*
233 / 147	-	*380*	*6,725*
149 / 94	-	*243*	*3,343*
65 / 41	*6.*	*106*	*1,193*
46 / 29	-	*75*	*387*
27 / 17	-	*44*	*147*
8 / 5	*4.*	*13*	*32*
5 / 3	-	*8*	*6*
2	*2.*	*3*	*1*

Table 1: The best rational upper approximations to ln*(3)*/ln*(2)*, the corresponding cycle length in *3x+1* problem and the bound for systematic computer verifications to eliminate this cycle length from the table.

We can see from the table that in order to eliminate the possible existence of a cycle of length *1,027,712,276* we should verify that all integers up to 1.25208×10^{18} converge. The next possible value for cycle length would then be *17,026,679,261*.

The table has been constructed in following way. First of all we computed first 30 terms in the continued fraction expansion for ln*(3)*/ln*(2)*. The result is

[1; 1, 1, 2 ,2, 3, 1, 5, 2, 23, 2, 23, 2 2, 1, 1, 55, 1, 4, 3, 1, 1, 15, 1, 9, 2, 5, 7, 1, 1 ,4, 8].

From this expansion we get the 30. convergent for ln*(3)*/ln*(2)*, which is *683,381,996,816,440/431,166,034,846,567*. Starting from this we get the next best upper approximations one by one using our Mathematica program `NextBestUpper`. All convergents having even order number between 10 and 30 appear in our table. That's natural because from the theory of continued fraction expansions it is generally known that the convergents are best rational approximations for the studied real number. Anyway, as we see from our table, they are not the only best approximations to a real number. For computing the last column value we have set equality on upper bound $n/k=\ln(3+1/R)/\ln(2)$. Solving R we get $R = 1/(2^{n/k}-3)$. Because n/k is the best possible rational approximation for $\ln(3)/\ln(2)$ we have used at least *100* digit precision in the computations of these bounds.

We have studied the Collatz problem for positive initial values. If we want to do the same for negative values also, we can define another integer sequence by defining $b_i=-a_i$ for all positive integer values of i. It is now easy to see, that for all i we have $b_{i+1}=3b_i-1$ if b_{i+1} is odd and $b_{i+1}=b_i/2$ if b_{i+1} is even. If b_1 is positive integer, all the iterates b_i have positive integer values.

It is easy to see that our *3x-1* sequences have trivial cycles *{1, 2}*, *{5, 14, 7, 20, 10}* and *{17, 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272, 136, 68, 34, 17}*.

Analogously to Lemma 1 we can prove the following result for *3x-1* problem.

LEMMA 6: *Let us suppose that there exists an nontrivial cycle of length m in some 3x-1 sequence (with positive initial value). Let k be the number of 3x-1 operations and n the number of x/2 operations in the cycle (Hence k+n=m). Suppose that the smallest number in sequence is greater than some (with a computer reachable bound) number R. Then*

$$\frac{\ln(3-\dfrac{1}{R})}{\ln(2)} \le \frac{n}{k} < \frac{\ln(3)}{\ln(2)}.$$

The following table gives best rational lower approximations for $\ln(3)/\ln(2)$. As in table 1, if approximation is a continued fraction convergent its order number is given in the second column. The third column gives the sum of the numerator and denominator of the rational approximation. This is the length of representing cycle of Collatz sequence. The last column gives the bound to be reached by computer calculations or by theoretical studies to eliminate the possibility of cycle of this size to exist.

Rational approximation n/k	**Convergent number**	**Cycle length m = n + k**	**Computational bound $R = 1/(3-2^{n/k})$**
83,130,157,078,217 / 52,449,289,519,716	*29.*	*135,579,446,597,933*	*1.20960 x 10^{28}*
64,789,416,887,513 / 40,877,570,830,877	-	*105,666,987,718,390*	*9.50064 x 10^{26}*
46,448,676,696,809 / 29,305,852,142,038	-	*75,754,528,838,847*	*3.58630 x 10^{26}*
28,107,936,506,105 / 17,734,133,453,199	-	*45,842,069,959,304*	*1.47286 x 10^{26}*
9,767,196,315,401 / 6,162,414,764,360	*27.*	*15,929,611,079,761*	*3.87339 x 10^{25}*
1,193,652,440,098 / 753,110,839,881	*25.*	*1,946,763,279,979*	*2.11025 x 10^{24}*
975,675,645,481 / 615,582,794,569	-	*1,591,258,440,050*	*2.01643 x 10^{23}*
757,698,850,864 / 478,054,749,257	-	*1,235,753,600,121*	*8.31572 x 10^{22}*
539,722,056,247 / 340,526,703,945	-	*880,248,760,192*	*4.03240 x 10^{22}*
321,745,261,630 / 202,998,658,633	-	*524,743,920,263*	*1.82213 x 10^{22}*

103,768,467,013 / 65,470,613,321	*23.*	*169,239,080,334*	*4.73167 x 10^{21}*
93,328,606,422 / 5,888,379,4651	-	*152,212,401,073*	*1.33203 x 10^{21}*
82,888,745,831 / 52,296,975,981	-	*135,185,721,812*	*7.01263 x 10^{20}*
72,448,885,240 / 45,710,157,311	-	*118,159,042,551*	*4.35564 x 10^{20}*
62,009,024,649 / 39,123,338,641	-	*101,132,363,290*	*2.89130 x 10^{20}*
51,569,164,058 / 32,536,519,971	-	*84,105,684,029*	*1.96378 x 10^{20}*
41,129,303,467 / 25,949,701,301	-	*67,079,004,768*	*1.32361 x 10^{20}*
30,689,442,876 / 19,362,882,631	-	*50,052,325,507*	*8.55167 x 10^{19}*
20,249,582,285 / 12,776,063,961	-	*33,025,646,246*	*4.97527 x 10^{19}*
9,809,721,694 / 6,189,245,291	*21.*	*15,998,966,985*	*2.15533 x 10^{19}*
9,179,582,797 / 5,791,671,912	-	*14,971,254,709*	*9.57790 x 10^{18}*
8,549,443,900 / 5,394,098,533	-	*13,943,542,433*	*5.84903 x 10^{18}*
7,919,305,003 / 4,996,525,154	-	*12,915,830,157*	*4.03027 x 10^{18}*
7,289,166,106 / 4,598,951,775	-	*11,888,117,881*	*2.95319 x 10^{18}*
6,659,027,209 / 4,201,378,396	-	*10,860,405,605*	*2.24096 x 10^{18}*
6,028,888,312 / 3,803,805,017	-	*9,832,693,329*	*1.73505 x 10^{18}*
5,398,749,415 / 3,406,231,638	-	*8,804,981,053*	*1.35714 x 10^{18}*
4,768,610,518 / 3,008,658,259	-	*7,777,268,777*	*1.06411 x 10^{18}*
4,138,471,621 / 2,611,084,880	-	*6,749,556,501*	*8.30251 x 10^{17}*
3,508,332,724 / 2,213,511,501	-	*5,721,844,225*	*6.39288 x 10^{17}*
2,878,193,827 / 1,815,938,122	-	*4,694,131,949*	*4.80408 x 10^{17}*
2,248,054,930 / 1,418,364,743	-	*3,666,419,673*	*3.46152 x 10^{17}*
1,617,916,033 / 1,020,791,364	-	*2,638,707,397*	*2.31207 x 10^{17}*
987,777,136 / 623,217,985	-	*1,610,995,121*	*1.31687 x 10^{17}*
357,638,239 / 225,644,606	*19.*	*583,282,845*	*4.46811 x 10^{16}*
85,137,581 / 53,715,833	*17.*	*138,853,414*	*5.15618 x 10^{15}*
68,049,666 / 42,934,559	-	*110,984,225*	*9.12750 x 10^{14}*
50,961,751 / 32,153,285	-	*83,115,036*	*3.84335 x 10^{14}*
33,873,836 / 21,372,011	-	*55,245,847*	*1.77685 x 10^{14}*
16,785,921 / 10,590,737	*15.*	*27,376,658*	*6.74993 x 10^{13}*
16,483,927 / 10,400,200	-	*26,884,127*	*2.96789 x 10^{13}*
16,181,933 / 10,209,663	-	*26,391,596*	*1.87697 x 10^{13}*
15,879,939 / 10,019,126	-	*25,899,065*	*1.35859 x 10^{13}*
15,577,945 / 9,828,589	-	*25,406,534*	*1.05572 x 10^{13}*
15,275,951 / 9,638,052	-	*24,914,003*	*8.57086 x 10^{12}*
14,973,957 / 9,447,515	-	*24,421,472*	*7.16787 x 10^{12}*
14,671,963 / 9,256,978	-	*23,928,941*	*6.12412 x 10^{12}*
14,369,969 / 9,066,441	-	*23,436,410*	*5.31731 x 10^{12}*
14,067,975 / 8,875,904	-	*22,943,879*	*4.67497 x 10^{12}*
13,765,981 / 8,685,367	-	*22,451,348*	*4.15146 x 10^{12}*
13,463,987 / 8,494,830	-	*21,958,817*	*3.71660 x 10^{12}*
13,161,993 / 8,304,293	-	*21,466,286*	*3.34963 x 10^{12}*
12,859,999 / 8,113,756	-	*20,973,755*	*3.03580 x 10^{12}*
12,558,005 / 7,923,219	-	*20,481,224*	*2.76435 x 10^{12}*
12,256,011 / 7,732,682	-	*19,988,693*	*2.52724 x 10^{12}*
11,954,017 / 7,542,145	-	*19,496,162*	*2.31834 x 10^{12}*
11,652,023 / 7,351,608	-	*19,003,631*	*2.13289 x 10^{12}*
11,350,029 / 7,161,071	-	*18,511,100*	*1.96717 x 10^{12}*

11,048,035 / 6,970,534	-	*18,018,569*	*1.81817 x 10^{12}*
10746,041 / 6,779,997	-	*17,526,038*	*1.68349 x 10^{12}*
10,444047 / 6,589,460	-	*17,033,507*	*1.56116 x 10^{12}*
10,142,053 / 6,398,923	-	*16,540,976*	*1.44956 x 10^{12}*
9,840,059 / 6,208,386	-	*16,048,445*	*1.34733 x 10^{12}*
9,538,065 / 6,017,849	-	*15,555,914*	*1.25335 x 10^{12}*
9,236,071 / 5,827,312	-	*15,063,383*	*1.16664 x 10^{12}*
8,934,077 / 5,636,775	-	*14,570,852*	*1.08641 x 10^{12}*
8,632,083 / 5,446,238	-	*14,078,321*	*1.01194 x 10^{12}*
8,330,089 / 5,255,701	-	*13,585,790*	*9.42637 x 10^{11}*
8,028,095 / 5,065,164	-	*13,093,259*	*8.77989 x 10^{11}*
7,726,101 / 4,874,627	-	*12,600,728*	*8.17537 x 10^{11}*
7,424,107 / 4,684,090	-	*12,108,197*	*7.60887 x 10^{11}*
7,122,113 / 4,493,553	-	*11,615,666*	*7.07689 x 10^{11}*
6,820,119 / 4,303,016	-	*11,123,135*	*6.57638 x 10^{11}*
6,518,125 / 4,112,479	-	*10,630,604*	*6.10462 x 10^{11}*
6,216,131 / 3,921,942	-	*10,138,073*	*5.65922 x 10^{11}*
5,914,137 / 3,731,405	-	*9,645,542*	*5.23801 x 10^{11}*
5,612,143 / 3,540,868	-	*9,153,011*	*4.83908 x 10^{11}*
5,310,149 / 3,350,331	-	*8,660,480*	*4.46072 x 10^{11}*
5,008,155 / 3,159,794	-	*8,167,949*	*4.10135 x 10^{11}*
4,706,161 / 2,969,257	-	*7,675,418*	*3.75960 x 10^{11}*
4,404,167 / 2,778,720	-	*7,182,887*	*3.43420 x 10^{11}*
4,102,173 / 2,588,183	-	*6,690,356*	*3.12400 x 10^{11}*
3,800,179 / 2,397,646	-	*6,197,825*	*2.82796 x 10^{11}*
3,498,185 / 2,207,109	-	*5,705,294*	*2.54513 x 10^{11}*
3,196,191 / 2,016,572	-	*5,212,763*	*2.27466 x 10^{11}*
2,894,197 / 1,826,035	-	*4,720,232*	*2.01573 x 10^{11}*
2,592,203 / 1,635,498	-	*4,227,701*	*1.76764 x 10^{11}*
2,290,209 / 1,444,961	-	*3,735,170*	*1.52971 x 10^{11}*
1,988,215 / 1,254,424	-	*3,242,639*	*1.30134 x 10^{11}*
1,686,221 / 1,063,887	-	*2,750,108*	*1.08196 x 10^{11}*
1,384,227 / 873,350	-	*2,257,577*	*8.71037 x 10^{10}*
1,082,233 / 682,813	-	*1,765,046*	*6.68110 x 10^{10}*
780,239 / 492,276	-	*1,272,515*	*4.72725 x 10^{10}*
478,245 / 301,739	-	*779,984*	*2.84469 x 10^{10}*
176,251 / 111,202	*13.*	*287,453*	*1.02959 x 10^{10}*
50,508 / 31,867	*11.*	*82,375*	*1.46214 x 10^{9}*
25,781 / 16,266	-	*42,047*	*2.12966 x 10^{8}*
1,054 / 665	*9.*	*1,719*	*5.07780 x 10^{6}*
569 / 359	-	*928*	*112,270*
84 / 53	*7.*	*137*	*8,461*
19 / 12	*5.*	*31*	*296*
11 / 7	-	*18*	*36*
3 / 2	*3.*	*5*	*6*
1	*1.*	*2*	*1*

Table 2: The best lower rational approximations to ln*(3)/*ln*(2)*, the corresponding cycle length in *3x-1* problem and the bound for systematic computer verifications to eliminate this cycle length from the table.

This table has been constructed in a way similar to table 1. First of all we take first 29 terms in the continued fraction expansion for ln*(3)/*ln*(2)*. The result is

[1; 1, 1, 2 ,2, 3, 1, 5, 2, 23, 2, 23, 2 2, 1, 1, 55, 1, 4, 3, 1, 1, 15, 1, 9, 2, 5, 7, 1, 1 ,4].

From this expansion we get the 29. convergent for ln*(3)/*ln*(2)*, which is *83,130,157,078,217/52,449,289,519,716*. Starting from this we get the next best lower approximations one by one using our Mathematica program `NextBestLower`. All convergents having odd order number less than or equal to 29 appear in our table. That's natural because of from the theory of continued fraction expansions it is generally known that the convergents are best rational approximations for the studied real number. Anyway, as we see from our table, they are not the only best approximations to a real number. In order to compute the last column value we have set equality on lower bound $n/k=\ln(3-1/R)/\ln(2)$. Solving R from this we get $R = 1/(3-2^{n/k})$. Because n/k is the best possible rational approximation for ln*(3)/*ln*(2)* we have used used at least *100* digit precision in the computations of these bounds.

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